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The Generalization of Chern-Simons Current and the Topological Tensor Current of p-Branes

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Abstract To make the gauge field theory foundation of the topological current of p-branes introduced in our previous work, we present a novel topological tensor current in $SO(N)$ gauge field theory. This non-Abelian gauge field tensor current is the straightforward generalization of the Chern-Simons topological current of strings. By making use of the $SO(N)$ gauge potential decomposition theory and the ϕ-mapping topological current theory, it is proved that the p-brane is created at every isolated zero of the Clifford vector field $\vec{\phi}(x)$ and the charges carried by p-branes are topologically quantized and labelled by the winding number of the ϕ-mapping.

Keywords Chern-Simons theory · Gauge field theory · p-branes · Topological current · Topological defects

1 Introduction

A new idea that we may live on a brane, namely the so-called 'brane-world' scenario of cosmology based on string/M-theory, has been largely discussed in the last several years. Dp-branes, and for more general cases, p-branes, which are extended objects embedded in a higher dimensional bulk spacetime, have been found to play important roles in revealing the nonperturbative structures of the modern string/M-theory [1–5] and so that received much attention. In recent years, the study of p-branes are also of importance in understanding string duality, M-theory unification and small distance structure of space (space-time). As they have also been proved to be topological defects in gauge theory [6–8], we can study them by using some proper topological methods. The starting point of this paper is to find a gauge field theory foundation of the ϕ-mapping topological current of p-branes introduced in our previous work [9]. For this purpose we construct a novel topological current of p-branes in the $SO(N)$ non-Abelian gauge field theory, which is different from the usual

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generalized Kalb-Ramond gauge field theory [8, 12]. This is the straightforward generalization of the Chern-Simons topological current of strings in $U(1)$ gauge field theory.

The Chern-Simons field theory has a lot of interest in the past [13–15]. In our previous work [11], we introduced a topological current to study the inner structure of $(2+1)$-dimensional Abelian Chern-Simons theory. Using the same method, we constructed a topological invariant for the knotlike strings in Chern-Simons field theory and shown that this invariant is just the total sum of all the self-linking and linking numbers of the knots family [16, 17]. In this paper, based on the $SO(N)$ gauge field theory we give a high-dimensional generalization of the Chern-Simons topological current and prove that it behaves as a δ-function of the Clifford vector field $\overrightarrow{\phi}(x)$, and the zeros of $\overrightarrow{\phi}(x)$ are corresponding to the p-branes.

This paper is arranged as follows. In Sect. 2, using the $U(1)$ gauge potential decomposition and the ϕ-mapping topological current theory, it is revealed that the two-dimensional Chern-Simons topological tensor current in space-time is corresponding to the topological current of string world-sheet. In Sect. 3, we present a high dimensional generalized Chern-Simons topological tensor current in $SO(N)$ gauge field theory. Using the decomposition theory of $SO(N)$ gauge potential, it is shown that this generalized Chern-Simons current is just the topological tensor current of p-branes we introduced in [9].

2 Chern-Simons Topological Tensor Current

We have shown in previous work [11, 16, 18] that the Chern-Simons current in $U(1)$ gauge field theory can be expressed as the topological current of strings,which is a naturally conserved current. In any quantum field theory,a conserved current is a quantity J^μ such that $\partial_\mu J^\mu = 0$. A conservation current is said to be topological if $\partial_\mu J^\mu = 0$ holds without use of the equations of motion.

Let A_μ be a $U(1)$ gauge potential in 4-dimensional space-time, the corresponding $U(1)$ gauge field tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1}$$

and the covariant derivative of a complex scalar field ψ

$$\psi = \phi^1 + i\phi^2 \tag{2}$$

is expressed as

$$D_\mu \psi = \partial_\mu \psi + i A_\mu \psi. \tag{3}$$

Thus we can introduce a topological current

$$j^\mu = \frac{1}{4\pi} \in^{\mu\nu\lambda} F_{\nu\lambda}, \quad \mu = 1, 2, 3 \tag{4}$$

it is well-known as the Chern-Simons current [11], using the decomposition theory of $U(1)$ gauge potential and the ϕ-mapping topological current theory [18] we have proved that the Chern-Simons current (4) is just the topological current of static strings [16].

The main purpose of this section is to generalize the traditional Chern-Simons vector current (4) to the following general covariant tensor current

$$j^{\lambda\sigma} = \frac{1}{4\pi} \frac{\in^{\lambda\sigma\mu\nu}}{\sqrt{g}} F_{\mu\nu} \quad (\lambda, \sigma = 0, 1, 2, 3) \tag{5}$$

in 4-dimensional Riemannian space-time, where g is the determinant of metric tensor $g_{\mu\nu}$. And we study the inner topological structure of the above Chern-Simons tensor current. This is of great importance to construct the topological gauge field theory of p-branes.

It has been shown in [19], the $U(1)$ gauge potential can be decomposed in terms of 2-dimensional unit vector n^A $(A = 1, 2)$

$$A_\mu = \epsilon_{AB}\, n^A \partial_\mu n^B - \partial_\mu \theta \tag{6}$$

where

$$\begin{aligned} n^A &= \frac{\phi^A}{\|\phi\|} \quad (A = 1, 2), \\ \|\phi\|^2 &= \phi^A \phi^A = \psi^* \psi, \end{aligned} \tag{7}$$

and θ is a phase factor which contributes nothing to the field tensor $F_{\mu\nu}$. Substituting (6) into (1) we have

$$F_{\mu\nu} = 2\, \epsilon_{AB}\, \partial_\mu n^A \partial_\nu n^B \tag{8}$$

and the tensor current (5) becomes

$$j^{\lambda\sigma} = \frac{1}{2\pi} \frac{\epsilon^{\lambda\sigma\mu\nu}}{\sqrt{g}}\, \epsilon_{AB}\, \partial_\mu n^A \partial_\nu n^B. \tag{9}$$

It is obvious $\partial_\lambda j^{\lambda\sigma} = 0$. From (7) we have $\partial_\mu n^A = \frac{\partial_\mu \phi^A}{\|\phi\|} + \phi^A \partial_\mu \frac{1}{\|\phi\|}$, and note $\epsilon^{\lambda\sigma\mu\nu}\, \partial_\mu \partial_\nu n^B = 0$ the tensor current (9) can be written as

$$\begin{aligned} j^{\lambda\sigma} &= \frac{1}{2\pi} \frac{\epsilon^{\lambda\sigma\mu\nu}}{\sqrt{g}}\, \epsilon_{AB}\, \partial_\mu \left[n^A \partial_\nu n^B \right] \\ &= \frac{1}{2\pi} \frac{\epsilon^{\lambda\sigma\mu\nu}}{\sqrt{g}}\, \epsilon_{AB}\, \partial_\mu \left[\frac{\phi^A}{\|\phi\|^2} \partial_\nu \phi^B \right], \end{aligned} \tag{10}$$

then using $\frac{\partial}{\partial \phi^A}(\ln \|\phi\|) = \frac{\phi^A}{\|\phi\|^2}$ and $\partial_\mu = \partial_\mu \phi^C \frac{\partial}{\partial \phi^C}$ we find

$$j^{\lambda\sigma} = \frac{1}{2\pi} \frac{\epsilon^{\lambda\sigma\mu\nu}}{\sqrt{g}}\, \epsilon_{AB}\, \frac{\partial}{\partial \phi^C} \left[\frac{\partial}{\partial \phi^A} (\ln \|\phi\|) \right] \partial_\mu \phi^C \partial_\nu \phi^B. \tag{11}$$

Defining the Jacobian tensor $D^{\lambda\sigma}(\phi/x)$ as

$$\epsilon^{CB}\, D^{\lambda\sigma}\left(\frac{\phi}{x}\right) = \epsilon^{\lambda\sigma\mu\nu}\, \partial_\mu \phi^C \partial_\nu \phi^B, \tag{12}$$

and by making use of the Green function relation in ϕ-space: $\Delta_\phi (\ln \|\phi\|) = 2\pi \delta^2(\vec{\phi})$ (where $\Delta_\phi = \frac{\partial}{\partial \phi^A} \frac{\partial}{\partial \phi^A}$ is the Laplacian in ϕ-space), one can prove [20–22] that the Chern-Simons topological tensor current takes the form of a δ-like function

$$j^{\mu\nu} = \frac{1}{\sqrt{g}} \delta^2(\vec{\phi}) D^{\mu\nu}\left(\frac{\phi}{x}\right), \tag{13}$$

from the above we see that

$$j^{\mu\nu}\begin{cases} = 0, & \text{if and only if } \vec{\phi} \neq 0, \\ \neq 0, & \text{if and only if } \vec{\phi} = 0. \end{cases} \tag{14}$$

Suppose that for the system of equations

$$\phi^1(x) = 0, \qquad \phi^2(x) = 0, \tag{15}$$

there are l isolated zeros and when the zeros are regular points [23], i.e. the rank of the Jacobian matrix $[\partial_\mu \phi^A]$ is 2, the solutions of (15) can be expressed as

$$x^\mu = z_i^\mu(\sigma, \tau), \quad i = 1, \dots, l \tag{16}$$

where σ and τ are the parameters of the string world-sheet.

It is well-known that $SO(2)$ group is isomorphic to $U(1)$ group, so we can express the topological tensor current (5) by $SO(2)$ gauge field strength tensor. For it is convenient to extend this current to $SO(N)$ gauge theory.

In $SO(N)$ gauge theory, the covariant derivative of a $SO(N)$ vector ϕ^A is defined as

$$D_\mu \phi^A = \partial_\mu \phi^A - \omega_\mu^{AB} \phi^B \tag{17}$$

where the gauge potential ω_μ^{AB} is antisymmetric

$$\omega_\mu^{AB} = -\omega_\mu^{BA}, \tag{18}$$

and the gauge field tensor is given by

$$F_{\mu\nu}^{AB} = \partial_\mu \omega_\nu^{AB} - \partial_\nu \omega_\mu^{AB} - \omega_\mu^{AC} \omega_\nu^{CB} + \omega_\nu^{AC} \omega_\mu^{CB}. \tag{19}$$

In $SO(2)$ case, the gauge potential ω_μ^{AB} $(A, B = 1, 2)$ have only one independent component ω_μ^{12} $(\omega_\mu^{12} = -\omega_\mu^{21})$. The relationship between $U(1)$ and $SO(2)$ gauge potential is defined as

$$A_\mu = \omega_\mu^{12}, \quad \text{i.e.}$$
$$A_\mu = \frac{1}{2} \epsilon_{AB}\, \omega_\mu^{AB}, \tag{20}$$

and the $U(1)$ gauge field tensor $F_{\mu\nu}$ can be expressed in terms of $SO(2)$ field tensor $F_{\mu\nu}^{AB}$

$$F_{\mu\nu} = \frac{1}{2} \epsilon_{AB}\, F_{\mu\nu}^{AB}.$$

Then the Chern-Simons tensor topological current (5) in 4-dimensional Riemannian manifold should be expressed as

$$j^{\lambda\sigma} = \frac{1}{8\pi} \frac{\epsilon^{\lambda\sigma\mu\nu}}{\sqrt{g}} \epsilon_{AB}\, F_{\mu\nu}^{AB} \tag{21}$$

that is the topological tensor current of creating string's world-sheets in $SO(2)$ gauge theory [10]. In the next section, we will extend this current to the $SO(N)$ gauge theory, which is the topological current creating p-branes.

3 Generalized Chern-Simons Current and Non-Abelian Topological Gauge Field Theory of *p*-Branes

In our previous work [9], we have presented a topological tensor current of p-branes. In this section, we will study the gauge field theory foundation of this topological tensor current of p-branes, and show that this topological current is the generalized Chern-Simons topological tensor current in $SO(N)$ gauge theory, which is defined as the following form

$$j^{\lambda\lambda_1\cdots\lambda_p} = \frac{(-1)^{N/2}}{A(S^{N-1})(N-1)!2^N}\left(\frac{1}{\sqrt{g}}\right)\epsilon^{\lambda\lambda_1\cdots\lambda_p\mu_1\mu_2\cdots\mu_N}\epsilon_{A_1A_2\cdots A_N} \times F^{A_1A_2}_{\mu_1\mu_2}F^{A_2A_4}_{\mu_3\mu_4}\cdots F^{A_{N-1}A_N}_{\mu_{N-1}\mu_N} \quad (N>2) \tag{22}$$

where $A(S^{N-1}) = 2\pi^{\frac{N}{2}}/\Gamma(\frac{N}{2})$ is the surface area of the unit sphere S^{N-1}. We note that this definition only for $N>2$, when $N=2$ this tensor current (22) is corresponding to (21) with a negative sign.

Firstly, we will prove that this current is conserved. Since $\epsilon^{\lambda\lambda_1\cdots\lambda_p\mu_1\mu_2\cdots\mu_N}$ is totally antisymmetric, the Bianchi identity

$$D_\alpha F^{AB}_{\beta\gamma} + D_\beta F^{AB}_{\gamma\alpha} + D_\gamma F^{AB}_{\alpha\beta} = 0, \quad \alpha,\beta,\gamma = 1,\ldots,N \tag{23}$$

can be written as

$$\epsilon^{\lambda\lambda_1\cdots\lambda_p\mu_1\mu_2\cdots\mu_N} D_\lambda F^{A_iA_{i+1}}_{\mu_i\mu_{i+1}} = 0. \tag{24$'$}$$

Noting that the tensor current $j^{\lambda\lambda_1\cdots\lambda_p}$ does not have the indices $A_1\cdots A_N$, and from (24′) we can deduce that

$$D_\lambda(j^{\lambda\lambda_1\cdots\lambda_p}\sqrt{g}) = \partial_\lambda(j^{\lambda\lambda_1\cdots\lambda_p}\sqrt{g}) = 0$$

therefore, this tensor current is identically conserved

$$\nabla_\lambda j^{\lambda\lambda_1\cdots\lambda_p} = \frac{1}{\sqrt{g}}\partial_\lambda(\sqrt{g}\, j^{\lambda\lambda_1\cdots\lambda_p}) = 0.$$

We will prove this generalized Chern-Simons $SO(N)$ topological tensor current is the topological tensor current creating p-branes. Let X be a compact and oriented D-dimensional Riemannian manifold with metric tensor $g_{\mu\nu}$, and $P(\pi, X, G)$ be a principal bundle with the structure group $G = SO(N)$, N $(N<D)$ is even and $N>2$. Let γ_A $(A=1,2,\ldots,N)$ be the basis of Clifford algebra, which satisfy

$$\gamma_A\gamma_B + \gamma_B\gamma_A = 2\delta_{AB}I \quad (I \text{ is the unit matrix})$$

and the Clifford vector field $\phi(x)$ on X is expressed as

$$\phi = \phi^A\gamma_A, \quad A = 1,2,\ldots,N.$$

Using ϕ define a Clifford unit vector

$$n^A = \frac{\phi^A}{\|\phi\|} \quad (A=1,2,\ldots,N;\ \|\phi\| = \sqrt{\phi^A\phi^A}), \tag{24}$$

$$n = n^A\gamma_A, \quad n^2 = I \tag{25}$$

which leads to

$$dnn + ndn = 0. \tag{26}$$

The unit vector n is identified as a section of the sphere bundle $S^{N-1}(X)$ over X, that defines a Gauss mapping

$$n : X \to S^{N-1}.$$

In $SO(N)$ gauge field theory the covariant derivative one-form of n is given by

$$Dn = dn - [\omega, n] \tag{27}$$

where ω is $SO(N)$ Lie algebra valued spin connection one-form

$$\omega = \omega_\mu dx^\mu, \quad \omega_\mu = \frac{1}{2}\omega_\mu^{AB} I_{AB} \tag{28}$$

in which the Lie algebra $I_{AB} = \frac{1}{4}(\gamma_A\gamma_B - \gamma_B\gamma_A)$ are the generators of $SO(N)$ Lie group. And the gauge field strength tensor two-form defined as

$$F = d\omega - \omega \wedge \omega, \tag{29a}$$

$$F = \frac{1}{2}F_{\mu\nu} dx^\mu \wedge dx^\nu \tag{29b}$$

which is $SO(N)$ Lie algebra valued 2-forms

$$F = \frac{1}{2}F^{AB} I_{AB} = \frac{1}{4}F_{\mu\nu}^{AB} I_{AB} dx^\mu \wedge dx^\nu \tag{30}$$

and the explicit form of $F_{\mu\nu}^{AB}$ is expressed in (19).

In the decomposition theory of $SO(N)$ gauge potential [24, 25], the gauge potential ω can be decomposed as

$$\begin{aligned} \omega &= \frac{1}{2}dnn + \frac{1}{2}nDn + \frac{1}{2}J_n(\omega), \\ J_n(\omega) &= \omega + n\omega n \end{aligned} \tag{31}$$

we call unit vector n in the decomposed expression the principal unit vector, one can choose n satisfying the gauge parallel condition $Dn = 0$ [26, 27], which is something like $\nabla_\lambda g_{\mu\nu} = 0$ in Riemannian geometry. The gauge potential ω (31) can be divided into two parts a and b

$$\omega = \frac{1}{2}(a + b)$$

$$\text{where } a = dnn, \ b = J_n(\omega) \tag{32}$$

from (26) one find that the 1-form a satisfies the anti-commutative relation

$$\{a, n\} = an + na = 0 \tag{33}$$

and 1-form $b = \omega + n\omega n$ satisfies the commutative relation

$$[b, n] = bn - nb = 0. \tag{34}$$

In fact from (27) we see that b may be an arbitrary 1-form satisfying commutative relation (34).

It has been proved that using only one principal unit vector $n = n^A \gamma_A$, one cannot find a Lie algebra valued 1-form that commutative with n. To construct a complete decomposition theory of $SO(N)$ gauge potential, it is necessary to define a system of two perpendicular unit vectors n and k, for which

$$n^2 = I, \qquad k^2 = I \tag{35}$$

and

$$nk + kn = 0. \tag{36}$$

We call the type of 1-form like $a = dnn$ the simple 1-form, it is easy to see that by means of unit vectors n and k, there exist only four kinds of simple 1-form, i.e.

$$\begin{aligned} a(n,n) &= dnn, \qquad & b(n,k) &= dnk, \\ a(k,k) &= dkk, \qquad & b(k,n) &= dkn \end{aligned} \tag{37}$$

and follows from (26) and (36) that they satisfy

$$\begin{aligned} \{a(n,n), n\} &= 0, \qquad & [b(n,k), n] &= 0, \\ \{a(k,k), k\} &= 0, \qquad & [b(k,n), k] &= 0. \end{aligned} \tag{38}$$

Therefore, when n is chosen as principal unit vector, i.e. $a = dnn$, the 1-form b that commutative with n can be constructed as

$$b = b(n,k) = dnk.$$

This means the gauge potential ω is decomposed as

$$\omega = \frac{1}{2}dnn + \frac{1}{2}dnk. \tag{39}$$

While we choose k as the principal unit vector, the decomposition of $SO(N)$ gauge potential is written as

$$\omega = \frac{1}{2}dkk + \frac{1}{2}dkn \tag{40}$$

where $a = dkk$ and $b = dkn$ satisfy

$$\{a, k\} = 0, \qquad [b, k] = 0. \tag{41}$$

We find that (39) and (40) can simply rewrite as

$$\omega = \frac{1}{2}dn(n+k), \tag{42a}$$

$$\omega = \frac{1}{2}dk(k+n). \tag{42b}$$

For a universal decomposition theory of $SO(N)$ gauge potential, the inner structure of gauge potential ω should not depend on the choosing of the principal unit vector. Therefore, the decomposition expression (42a) and (42b) must be self-consistent locally, this means

$$dn(n+k) = dk(k+n). \tag{43}$$

Multiplying the above equation by $(n+k)$ from both right side and using (35) and (36), we have an important relation

$$dk = dn. \tag{44}$$

Substituting this relation into (40), we find a unique decomposition expression for $SO(N)$ gauge potential

$$\omega = \frac{1}{2}dnk + \frac{1}{2}dnn. \tag{45}$$

Using $dk = dn$ and the definitions (38), we find the following crossing relation

$$a(k,k) = b(n,k) \quad \text{and}$$
$$b(k,n) = a(n,n).$$

We must point out here that due to the existence of the above crossing relation, the decomposition expression (39) is unique since the 1-form $a(n,n)$ and $a(k,k)$ cannot be arbitrary.

Substituting the decomposition expression for gauge potential (39) into (29a) and noting $dk = dn$ the $SO(N)$ gauge field 2-form can be expressed as

$$F = -\frac{1}{2}dn \wedge dn = -\partial_\mu n^A \partial_\nu n^B I_{AB} dx^\mu \wedge dx^\nu. \tag{46}$$

Comparing expression (30) and (46), we get a concrete expression in decomposition theory

$$F^{AB}_{\mu\nu} = -4\partial_\mu n^A \partial_\nu n^B. \tag{47}$$

Substituting (47) into (22), the generalized Chern-Simons tensor current is expressed in the following form

$$j^{\lambda\lambda_1\cdots\lambda_p} = \frac{1}{A(S^{N-1})(N-1)!}\left(\frac{1}{\sqrt{g}}\right) \in^{\lambda\lambda_1\cdots\lambda_p\mu_1\mu_2\cdots\mu_N} \in_{A_1A_2\cdots A_N} \partial_{\mu_1} n^{A_1} \partial_{\mu_2} n^{A_2} \cdots \partial_{\mu_N} n^{A_N}, \quad (N > 2) \tag{48}$$

this is just the topological tensor current of p-branes we introduced in [9].

Here we must emphasize that as a special case of (48) when the dimension of the Riemannian manifold X and the dimension of the structure group G are equal, i.e. $D = N$ (even), a similar result as (48) was given by Chern [28–30] in the Gauss-Bonnet-Chern (GBC) theorem, whose instructive idea was to work on the sphere bundle $S(X)$. In GBC theorem the Euler density ρ is defined by the N-form

$$\Lambda = \frac{(-1)^{\frac{N}{2}}}{2^N \pi^{\frac{N}{2}}(N/2)!} \in_{A_1A_2\cdots A_{N-1}A_N} F^{A_1A_2} \wedge \cdots \wedge F^{A_{N-1}A_N} = \rho\sqrt{g}d^N x. \tag{49}$$

Using a recursion method and sphere bundle formulation, Chern proved [28–30] that the formulation of GBC form can be represented as

$$\Lambda = \frac{1}{A(S^{N-1})(N-1)!} \in_{A_1A_2\cdots A_N} dn^{A_1} \wedge dn^{A_2} \wedge \cdots \wedge dn^{A_N} \tag{50}$$

and the Euler density is expressed as

$$\rho = \frac{1}{A(S^{N-1})(N-1)!}\left(\frac{1}{\sqrt{g}}\right) \in^{\mu_1\mu_2\cdots\mu_N} \in_{A_1A_2\cdots A_N} \partial_{\mu_1} n^{A_1} \partial_{\mu_2} n^{A_2} \cdots \partial_{\mu_N} n^{A_N}. \tag{51}$$

Therefore the theory of tensor current of p-branes can be looked upon as the generalization of GBC theorem.

In the following we will study the topological structure of the generalized Chern-Simons tensor current (22) i.e. (48) and show that this is exactly the topological tensor current creating p-branes. From (24) we have the relations

$$\partial_\mu n^A = \frac{1}{\|\phi\|}\partial_\mu \phi^A + \phi^A \partial_\mu\left(\frac{1}{\|\phi\|}\right) \tag{52}$$

$$-\frac{1}{N-2}\frac{\partial}{\partial \phi^A}\left(\frac{1}{\|\phi\|^{N-2}}\right) = \frac{\phi^A}{\|\phi\|^N}. \tag{53}$$

Using the above relations, the topological tensor current (48) can be expressed in the following

$$\begin{aligned} j^{\lambda\lambda_1\cdots\lambda_p} &= \frac{1}{A(S^{N-1})(N-1)!}\left(\frac{1}{\sqrt{g}}\right) \in^{\lambda\lambda_1\cdots\lambda_p\mu_1\mu_2\cdots\mu_N} \in_{A_1A_2\cdots A_N} \partial_{\mu_1} \\ &\quad \times (n^{A_1} \partial_{\mu_2} n^{A_2} \cdots \partial_{\mu_N} n^{A_N}) \\ &= \frac{1}{A(S^{N-1})(N-1)!}\left(\frac{1}{\sqrt{g}}\right) \in^{\lambda\lambda_1\cdots\lambda_p\mu_1\mu_2\cdots\mu_N} \in_{A_1A_2\cdots A_N} \partial_{\mu_1} \\ &\quad \times \left(\frac{\phi^{A_1}}{\|\phi\|^N} \partial_{\mu_2}\phi^{A_2} \cdots \partial_{\mu_N}\phi^{A_N}\right) \\ &= -\frac{1}{A(S^{N-1})(N-1)!(N-2)}\left(\frac{1}{\sqrt{g}}\right) \in^{\lambda\lambda_1\cdots\lambda_p\mu_1\mu_2\cdots\mu_N} \in_{A_1A_2\cdots A_N} \\ &\quad \times \partial_{\mu_1}\left[\frac{\partial}{\partial\phi^{A_1}}\left(\frac{1}{\|\phi\|^{N-2}}\right)\right]\partial_{\mu_2}\phi^{A_2} \cdots \partial_{\mu_N}\phi^{A_N} \end{aligned} \tag{54}$$

and we have

$$\begin{aligned} j^{\lambda\lambda_1\cdots\lambda_p} &= C_N\left(\frac{1}{\sqrt{g}}\right) \in^{\lambda\lambda_1\cdots\lambda_p\mu_1\mu_2\cdots\mu_N} \in_{A_1A_2\cdots A_N} \frac{\partial}{\partial\phi^A}\frac{\partial}{\partial\phi^{A_1}}\left(\frac{1}{\|\phi\|^{N-2}}\right) \\ &\quad \times \partial_{\mu_1}\phi^A \partial_{\mu_2}\phi^{A_2} \cdots \partial_{\mu_N}\phi^{A_N} \end{aligned} \tag{55}$$

where $C_N = -\frac{1}{A(S^{N-1})(N-1)!(N-2)}$. Defining the Jacobian tensor $D^{\lambda\lambda_1\cdots\lambda_p}(\frac{\phi}{x})$ as follows

$$\in^{AA_2\cdots A_N} D^{\lambda\lambda_1\cdots\lambda_p}\left(\frac{\phi}{x}\right) = \in^{\lambda\lambda_1\cdots\lambda_p\mu_1\mu_2\cdots\mu_N} \partial_{\mu_1}\phi^A \partial_{\mu_2}\phi^{A_2} \cdots \partial_{\mu_N}\phi^{A_N} \tag{56}$$

and using the generalized Laplacian Green function relation in ϕ-space

$$\Delta_\phi\left(\frac{1}{\|\phi\|^{N-2}}\right) = -\frac{4\pi^{\frac{N}{2}}}{\Gamma(N/2-1)}\delta(\vec{\phi}) \tag{57}$$

where $\Delta_\phi = \frac{\partial^2}{\partial\phi^A\partial\phi^A}$ is the N-dimensional Laplacian operator in ϕ-space, we obtain a δ-function like tensor current

$$j^{\lambda\lambda_1\cdots\lambda_p} = \delta(\vec{\phi}) D^{\lambda\lambda_1\cdots\lambda_p}\left(\frac{\phi}{x}\right)\left(\frac{1}{\sqrt{g}}\right). \tag{58}$$

We find that $j^{\lambda\lambda_1\cdots\lambda p} \neq 0$ only when $\vec{\phi}(x) = 0$

$$\phi^A(x) = 0, \quad A = 1, \ldots, N. \tag{59}$$

Suppose that the vector field $\vec{\phi}(x)$ possesses l isolated zeros, according to the implicit function theorem [23], when the zeros are regular points of $\vec{\phi}(x)$, i.e. the rank of the Jacobian matrix $[\partial_\lambda\phi^A]$ is N, the solution of $\vec{\phi}(x) = 0$ can be parameterized by

$$x^\mu = z_i^\mu(u^1, u^2, \ldots, u^{D-N}), \quad i = 1, \ldots, l, \tag{60}$$

where the subscript i represents the i-th solution and the parameters $u = u(u^1, \ldots, u^{D-N})$ span a $(D-N)$-dimensional submanifold of X, denoted by M_i, which corresponds to a p-brane $(p = D - N - 1)$ with spatial p dimension and M_i is it's worldvolume. We see that the tensor current $j^{\lambda\lambda_1\cdots\lambda p}$ is not vanished only on the worldvolume manifolds M_i $(i = 1, \ldots, l)$. Therefore, the generalized Chern-Simons tensor current is indeed the topological current creating p-branes. Here, we must point out that in the above theory the p-branes can be considered as topological defects [6–8], in this case for our theory the Clifford vector field $\phi^A(x)$ $(A = 1, \ldots, N)$ can be looked upon as the generalized order parameters [32–34] for p-branes.

In the end of this section we briefly discuss the inner structure of the topological tensor current $j^{\lambda\lambda_1\cdots\lambda_p}$.The detailed discussion can be found in [9]. According to the ϕ-mapping theory [9, 11, 18, 20–22] and the δ-function theory [35], the $\delta(\vec{\phi}(x))$ in (58) can be expanded on the singular submanifolds M_i $(i = 1, \ldots, l)$ as

$$\delta(\vec{\phi}(x)) = \sum_{i=1}^{l} W_i \int_{M_i} \delta^D(x - z_i(u))\sqrt{g_u}d^{(D-N)}u \tag{61}$$

where W_i is the winding number of Gauss map $n : X \to S^{N-1}$ and g_μ is the determinant of the metric on M_i. So the topological current of the p-branes (58) can be expressed explicitly as

$$j^{\lambda\lambda_1\cdots\lambda_p} = \left(\frac{1}{\sqrt{g}}\right) D^{\lambda\lambda_1\cdots\lambda_p}\left(\frac{\phi}{x}\right)\sum_{i=1}^{l} W_i \int_{M_i} \delta^D(x - z_i(u))\sqrt{g_u}d^{(D-N)}u. \tag{62}$$

Therefore the generalized Chern-Simons topological tensor current (22) expressed in terms of the $SO(N)$ gauge field tensor $F_{\mu\nu}^{AB}$ has the profound physics content of creating p-branes, and the winding number W_i is looked upon as the topological charge carried by the i-th p-branes.

4 Conclusion

The above formulation and rigorous proof indicate that the $SO(N)$ non-Abelian gauge field topological tensor current constructed in this paper is a novel development of the theories of

p-branes, and is of great importance. This theory is an important generalization of our $U(1)$ topological current theory of strings to the theory of p-branes. And the theory is based on the decomposition theory of $SO(N)$ gauge potential [24, 25, 31] and the ϕ-mapping topological current theory [9, 18] we suggested.

The $SO(N)$ gauge field tensor current theory in this paper is only for the case that N is even, the theory in the case of odd N will be discussed elsewhere.

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